by.

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Que 1:- State and Prove Cauchy's root test theorem.

Let u_n be always positive and $\lim_{n\to\infty} u_n^{1/n} = l$

then (i) If $l<1$, $\sum u_n$ is Convergent

(ii) If $l>1$, $\sum u_n$ is divergent

(iii) If $l=1$, the test fails and the series may be either Convergent or divergent.

Proof:- Let $l<1$

Since $l<1$, We can choose and $\epsilon>0$ such that

$l+\epsilon<1$

Let $l+\epsilon=r$ so that $0<r<1$

Since $\lim_{n\to\infty} u_n^{1/n} = l$ therefore there exists positive integer m such that

$$\left| u_n^{1/n} - l \right| < \epsilon \quad \text{When } n \geq m.$$

i.e $l-\epsilon < u_n^{1/n} < l+\epsilon$ When $n \geq m$

i.e $(l-\epsilon)^n < u_n < (l+\epsilon)^n$ when $n \geq m$

Taking the last inequality We find that

$u_n < r^n$ Where $r = l+\epsilon < 1$ for all $n \geq m$.

But the series $r^m + r^{m+1} + \cdots$ is a series in G.P. whose Common ratio $r<1$ and hence it is Convergent. Therefore by the Comparison test $u_m + u_{m+1} + \cdots$ is Convergent

Hence $\sum u_n$ is Convergent

(ii) Let $l>1$

Since $l>1$, We can choose ~~and~~ an $\epsilon>0$ such that

$l-\epsilon>1$

Let $l-\epsilon=R$ so that $R>1$

Since $\lim_{n\to\infty} u_n^{1/n} = 1$ therefore there exists a positive integer m such that

$$\left| u_n^{1/n} - l \right| < \epsilon \quad \text{when } n \geq m$$

i.e $\quad l - \epsilon < u_n^{1/n} < 1 + \epsilon \quad$ when $n \geq m$

i.e $\quad (1-\epsilon)^n < u_n < (1+\epsilon)^n$ when $n \geq m$

Taking the first inequality, we find that

$$u_n > R^n \text{ where } R = l - \epsilon > 1 \text{ for all } n \geq m.$$

Thus we find that u_n does not tend to zero. Hence Σu_n is not convergent. But a series of positive terms must either converge or diverge and so Σu_n is divergent.

Case III. We will illustrate this case when $l = 1$ by taking suitable examples.

Let $u_n = \frac{1}{n}$ then $u_n^{1/n} = \left(\frac{1}{n}\right)^{1/n} = \frac{1}{n^{1/n}}$

$$\therefore \lim_{n\to\infty} u_n^{1/n} = \frac{1}{\lim_{n\to\infty} n^{1/n}} = \frac{1}{1} = 1$$

But $\Sigma \frac{1}{n}$ is a divergent series

Thus we see in this case that when $\lim u_n^{1/n} = 1$ the series Σu_n is divergent

on the other hand let $u_n = \frac{1}{n^2}$

then in this case also $u_n^{1/n} = \frac{1}{n^{2/n}} = \frac{1}{(n^{1/n})^2}$

$$\therefore \lim_{n\to\infty} u_n^{1/n} = 1$$

But the series $\Sigma \frac{1}{n^2}$ is a convergent series

Thus we see that in this case when $\lim_{n\to\infty} u_n^{1/n} = 1$ the series Σu_n is convergent.

These two examples show that if $\lim_{n\to\infty} u_n^{1/n} = 1$ the series Σu_n may be either convergent or divergent that is the test fails